

Gradient estimates and symmetrization for Fisher-KPP front propagation with fractional diffusion

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Abstract

In this paper, we study gradient decay estimates for solutions to the multi-dimensional Fisher-KPP equation with fractional diffusion. It is known that this equation exhibits exponentially advancing level sets with strong qualitative upper and lower bounds on the solution. However, little has been shown concerning the gradient of the solution. We prove that, under mild conditions on the initial data, the first and second derivatives of the solution obey a comparative exponential decay in time. We then use this estimate to prove a symmetrization result, which shows that the reaction front flattens and quantifiably circularizes, losing its initial structure.

Keywords: Fisher-KPP equations, fractional diffusion, gradient decay estimates, traveling fronts.

1 Introduction

The goal of this paper is to understand a strong symmetrisation phenomenon, observed in [7], for the level sets of reaction-diffusion equations of Fisher-KPP type with fractional diffusion. The model under consideration is

$$\partial_t u + \Lambda^\alpha u = f(u) \tag{1.1}$$

on \mathbb{R}^d . Here, $\alpha \in (0, 2)$ and Λ^α is the fractional Laplacian of order $\alpha/2$:

$$(-\Delta)^{\alpha/2} u(x) = \mathcal{F}^{-1}(|\xi|^\alpha \hat{u}(\xi))(x) = \lim_{\varepsilon \rightarrow 0} c_{N,\alpha} \int_{|y| > \varepsilon} \frac{u(x) - u(x+y)}{|y|^{d+\alpha}} dy.$$

The nonlinear term f is assumed to be of KPP-type [10]: $f(0) = f(1) = 0$, $f(y) > 0$ for $y \in (0, 1)$, and $f'(y) < f(y)/y$. Notation-wise, let $\kappa = f'(0)$ and $\lambda = \kappa/(d + \alpha)$.

Under these assumptions, and for a compactly supported initial datum $u(., 0) := u_0$, it is well-known that the level sets of u will spread exponentially fast in time (Cabr  -Roquejoffre [5]):

Theorem 1.1 *Under the above assumptions,*

- For all $c > \lambda$, we have $\lim_{t \rightarrow +\infty} \sup_{|x| \geq e^{ct}} u(x, t) = 0$.
- For all $c < \lambda$, we have $\lim_{t \rightarrow +\infty} \inf_{|x| \leq e^{ct}} u(x, t) = 1$.

Thus, when renormalized by the exponential, the level sets of u asymptotically look round. It is then natural to ask whether this property holds in a more precise fashion, and a first answer is that given in Cabr  -Coulon-Roquejoffre [6]: for a given $h \in (0, 1)$ we have

$$\{u = h\} \subseteq \{C^{-1}e^{\lambda t} \leq |x| \leq Ce^{\lambda t}\},$$

for some constant $C > 0$. The next question is whether one can make this constant more precise. Given the rapid growth of the level sets, one could expect the possibility for very erratic behavior. To confirm this, the case was investigated numerically by A.-C. Coulon in her PhD thesis. We reproduce here a sample of her simulations. The initial datum (Fig. 1) is pictured below,

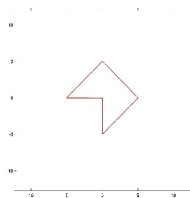


Figure 1: Level sets of u_0

and the time evolution is shown in Fig. 2. Surprisingly, this strongly suggests

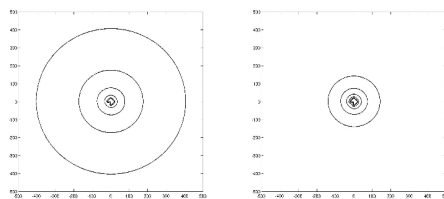


Figure 2: Level sets of u : $\alpha = 1$ (left) and $\alpha = 1.6$ (right)

asymptotic symmetrization. So we may ask whether a very general theorem of Jones [9] applies. It concerns the solutions of (1.1) with $\alpha = 2$:

$$\partial_t u - \Delta u = f(u) \quad (1.2)$$

with any nonlinearity f - not limited to KPP. The theorem asserts that, if u_0 is compactly supported, then, for any later time, and for any regular value h of u , and for any x on the level set $\{u(\cdot, t) = h\}$, the normal line $\Delta_{x,t}$ to the level set passing through x intersects the convex hull of $\text{supp } u_0$. If the level sets of u expand, this has very important consequences: the level sets of u symmetrize asymptotically, and have bounded oscillation. This is a very remarkable result, one reason being that it does not depend on the precise expansion rate of the level sets of u . In particular, if the nonlinearity f is of KPP type, then

$$\{u = h\} = \{|x| = 2\sqrt{\kappa t} - \frac{d+2}{2}\text{Log}t + O(1)\}. \quad (1.3)$$

The term $2\sqrt{\kappa t}$ is due to Aronson-Weinberger [1], and the logarithmic correction to Bramson [4] ($d = 1$), and Gärtner [8] ($d > 1$).

Let us briefly describe a proof, due to H. Berestycki [2], of Jones's theorem: assume the contrary, then there is a hyperplane H containing $\Delta_{x,t}$ and such that the convex hull of $\text{supp } u_0$ lies strictly in H^- , the lower half space bounded by H . If \hat{u} is the reflection of $u(\cdot, t)$ about H , and $v = u - \hat{u}$, then v satisfies a linear equation and is positive in H^- , since it is positive at $t = 0$ and vanishes on H . At time t , any derivative of v in a direction normal to H is nonzero, contradicting that $\Delta_{x,t} \subset H$. We immediately see that this argument cannot be applied to our case, simply because v would need to be nonnegative in H^+ instead of H^- , which is impossible.

We are going to show that, nevertheless, a rather strong form of symmetrization occurs. The ingredient is the following gradient estimate for u ; we believe that it is of general interest.

Theorem 1.2 *Assume $u(x, 0) := u_0(x)$ to be continuous, nonnegative and nonzero, and $u_0(x) = O(e^{-\varepsilon|x|})$ as $|x| \rightarrow +\infty$, for some $\varepsilon > 0$. Then we have, for a solution $u(x, t)$ of (1.1), a universal constant C , and a $\delta > 0$ (depending on λ and α):*

$$|\nabla u(x, t)| \leq C e^{-\delta t} u(x, t), \quad (1.4)$$

and

$$|\nabla^2 u(x, t)| \leq C e^{-\delta t} u(x, t). \quad (1.5)$$

Theorem 1.2 allows us to prove an analogous estimate for the fractional Laplacian. This permits us in turn to reduce the problem to the ODE $\dot{u} = (1 + O(e^{-\delta t}))f(u)$, which is much simpler to study. The main result of the paper is

Theorem 1.3 *Let u_0 be as in Theorem 1.2. For every $h \in (0, 1)$, there is a constant $q_h(u_0) > 0$ and $\delta > 0$ such that*

$$\{u = h\} = \{|x| = q_h(u_0) e^{\lambda t} (1 + O(e^{-\delta t}))\}. \quad (1.6)$$

This provides an explanation to the observed behavior of u . However we note that our results could be improved in two ways.

- The assumptions on u_0 seem slightly non optimal: indeed, it would certainly suffice to have $u_0(x) = O(|x|^{d+\alpha+\varepsilon})$ with $\varepsilon > 0$. Our theorem would remain valid under that assumption, at the cost of heavier computations. However, when u_0 decays like (or slower than) $|x|^{d+\alpha}$ different phenomena occur, as was observed in [7].
- We do not go as far as proving a Jones type theorem. Indeed what we obtain is a precise estimate of the normalized level sets instead of the true level sets. In other words, lower order (but still exponentially growing) terms may prevent a full symmetrization.

These issues will be investigated in a future paper.

In Section 2, we gather some known (but useful) facts that will be used throughout the proofs of Theorems 1.2 and 1.3. Section 3 is devoted to the proof of Theorem 1.2, while Section 4 is devoted to estimating the fractional Laplacian. Theorem 1.3 is then proved in Section 5.

2 Preliminary material

The gradient estimate (1.4) will be obtained by examining its representation in three different ranges, which is reflected by the collection of results below, that we recall for the reader's convenience. The proof for the estimate on second derivatives (1.5) follows a similar approach, but will itself make use of (1.4).

2.1 Invariant coordinates

The starting point is the following estimate, proved in [6]

Theorem 2.1 *We have, for a universal $C > 0$:*

$$\frac{C^{-1}}{1 + e^{-\kappa t} |x|^{d+\alpha}} \leq u(x, t) \leq \frac{C}{1 + e^{-\kappa t} |x|^{d+\alpha}}. \quad (2.1)$$

This motivates the introduction of the invariant coordinates $\xi = xe^{-\lambda t}$:

$$\partial_t u - \lambda \xi \cdot \nabla_\xi u + e^{-\alpha \lambda t} \Lambda^\alpha u - f(u) = 0. \quad (2.2)$$

For any fixed coordinate x_i , set $\phi = \partial_{x_i} u$. We then have

$$\partial_t \phi + \Lambda^\alpha \phi = f'(u) \phi. \quad (2.3)$$

Letting $v = \partial_{\xi_i} u = e^{\lambda t} \phi$ yields

$$\partial_t v - \lambda \xi \cdot \nabla v + e^{-\alpha \lambda t} \Lambda^\alpha v - f'(u) v - \lambda v = 0. \quad (2.4)$$

For another fixed coordinate x_j , set $\psi = \partial_{x_j}\phi = \partial_{x_j}\partial_{x_i}u$ and $\bar{\phi} = \partial_{x_j}u$, with $\bar{v} = e^{\lambda t}\bar{\phi}$. This then gives us

$$\partial_t\psi + \Lambda^\alpha\psi = f''(u)\phi\bar{\phi} + f'(u)\psi, \quad (2.5)$$

along with the associated equation in exponential coordinates: for $V = \partial_{\xi_j}\partial_{\xi_i}u$, we have

$$\partial_t V - \lambda\xi \cdot \nabla V + e^{-\alpha\lambda t}\Lambda^\alpha V - f'(u)V - f''(u)v\bar{v} - 2\lambda V = 0. \quad (2.6)$$

A consequence of Theorem 1.3 is the large time convergence of the initially compactly supported (or initially exponentially decreasing) solutions of (2.2) to a steady profile. Notice indeed that the steady equation

$$-\lambda\xi \cdot \nabla_\xi u = f(u) \quad (2.7)$$

has a unique one-parameter family of radial solutions $(u_\tau(|\xi|))_{\tau>0}$. Then, assuming Theorem 1.3:

Theorem 2.2 *There exists a $\tau_\infty(u_0) > 0$ such that*

$$\lim_{t \rightarrow +\infty} u(\xi, t) = u_{\tau_\infty(u_0)}(\xi),$$

uniformly on compact subsets of \mathbb{R}^d .

2.2 Heat kernel

Let $\rho_\alpha(x, t)$ be the heat kernel of Λ^α , in other words the solution of $\rho_t + \Lambda^\alpha\rho = 0$ having, as initial datum, the Dirac mass at 0.

Proposition 2.3 *There is $C > 0$ such that, for large t and x , we have $\rho_\alpha(x, t) = t^{-d/\alpha}p_\alpha(x/t^{d/\alpha})$, with*

$$\frac{C^{-1}}{|\zeta|^{d+\alpha+1}} \leq |\nabla p_\alpha(\zeta)| \leq \frac{C}{|\zeta|^{d+\alpha+1}}, \quad (2.8)$$

and there is $c_\alpha > 0$, $\delta > 0$ such that

$$\frac{C^{-1}}{|\zeta|^{d+\alpha+\delta}} \leq p_\alpha(\zeta) - \frac{c_\alpha}{|\zeta|^{d+\alpha}} \leq \frac{C}{|\zeta|^{d+\alpha+\delta}}. \quad (2.9)$$

It is important to note that the estimates (2.8) and (2.9) are intended for ζ bounded away from 0; ρ_α and its derivatives are bounded functions (in x) for any $t > 0$. A standard way to prove the above proposition is to write

$$\rho_\alpha(x, t) = \mathcal{F}^{-1} \left(e^{-|\xi|^\alpha t} \right),$$

and to evaluate the inverse Fourier transform with the aid of Polya integrals; see for instance [11].

2.3 Comparison principles

We will also require the following easy extension of the Maximum Principle:

Theorem 2.4 (Selective Comparison Principle) *Let $\Omega(t)$ be a time-dependent family of compact domains in \mathbb{R}^d (in the ξ variable) with smooth boundaries and continuous time dependence; that is, $\{(\xi, t) \mid \xi \in \Omega(t)\}$ is an open set in $\mathbb{R}^d \times \mathbb{R}_+$. Let $v(\xi, t)$ be the solution to (2.4) and let $w(\xi, t)$ be a positive function such that $w > |v|$ at $t = 0$. If, for all $t > 0$ we have $w > |v|$ on the closure of the complement of $\Omega(t)$ and*

$$\partial_t w - \lambda \xi \cdot \nabla w + e^{-\alpha \lambda t} \Lambda^\alpha w - f'(u)w - \lambda w > 0$$

everywhere inside $\Omega(t)$, then $w(\xi, t) \geq |v(\xi, t)|$ on all of $\mathbb{R}^d \times \mathbb{R}_+$.

PROOF. We first show that $w > v$ by looking at the equation for $q = w - v$ on $\Omega(t)$ which, by assumption, starts positive at $t = 0$. Observe that, if q achieves a global minimum value of 0 at time \bar{t} and location $\bar{\xi} \in \Omega(\bar{t})$, then we easily have $\partial_t q(\bar{\xi}, \bar{t}) > 0$ (all other terms on the left are nonpositive); this, however, crucially requires that $w > v$ outside $\Omega(\bar{t})$. So, by continuity in time, q can never become negative inside $\Omega(t)$. To complete the proof, we need to show that $w > -v$. But this is done in precisely the same manner as before, now examining the equation for $\bar{q} = w + v$ and showing that it too remains positive for all time. \bullet

Also note that the above result (with a minor modification) also holds for a solution V of (2.6). That is, if W is a positive function such that $W > |V|$ on the closure of the complement of $\Omega(t)$ for all $t > 0$ and also $W > |V|$ on $\Omega(0)$, and

$$\partial_t W - \lambda \xi \cdot \nabla W + e^{-\alpha \lambda t} \Lambda^\alpha W - f'(u)W - 2\lambda W > |f''(u)v\bar{v}|$$

everywhere inside $\Omega(t)$, then $W(\xi, t) > |V(\xi, t)|$ on all of $\mathbb{R}^d \times \mathbb{R}_+$. The proof is exactly the same since the equations for $W - V$ and $W + V$ have nonnegative inhomogeneities on the right side of the inequality.

Finally let us mention the following version of Kato's inequality and its well-known consequence for the Fisher-KPP equation.

Proposition 2.5 *If $u(x)$ is smooth, then*

$$\Lambda^\alpha |u(x)| \leq \text{sgn}(u) \Lambda^\alpha u(x),$$

in the distributional sense (and in the classical sense if $\alpha < 1$).

PROOF. Recall the elementary inequality $|a| - |b| \geq \text{sgn}(b)(a - b)$. This implies, for all $(x, h) \in \mathbb{R}^d \times \mathbb{R}^d$:

$$|u(x + h)| - |u(x)| \geq \text{sgn}(u(x))(u(x + h) - u(x)).$$

Integrating both sides in the variable h over $\mathbb{R}^d \setminus (-\varepsilon, \varepsilon)^d$ and letting $\varepsilon \rightarrow 0$ yields the result. \bullet

This implies the following lemma:

Lemma 2.6 *If $\phi(x, t)$ is either u_t , or $\partial_{x_i} u(x, t)$, ($i \in \{1, \dots, d\}$), where $u(x, t)$ is the solution of (1.1), then*

$$\partial_t |\phi| + \Lambda^\alpha |\phi| \leq \kappa |\phi|.$$

Additionally, we have

$$\partial_t |\psi| + \Lambda^\alpha |\psi| \leq |f''(u)\phi\bar{\phi}| + \kappa |\psi|$$

for $\psi(x, t)$ a solution to (2.5).

PROOF. Multiplying (2.3) by $\text{sgn}(\phi)$ (respectively (2.5) by $\text{sgn}(\psi)$) and using Proposition 2.5 yields the result. \bullet

3 The gradient decay estimate

Throughout this section, all inequalities will be up to a constant independent of the solutions. We also weaken our requirements on the decay of the initial data. Specifically, we insist that

$$0 \leq u_0(x) \leq \frac{C}{1 + |x|^{d+\alpha+1}}.$$

With the notations of Section 2, we would like to prove an exponential decay rate for the gradient (1.4) and second derivatives (1.5) of the form

$$|\phi(x, t)| \leq u(x, t)e^{-\delta t} \quad \text{and} \quad |\psi(x, t)| \leq u(x, t)e^{-\delta t}. \quad (3.1)$$

To do this, we will need to examine both representations for the evolution of the derivative ((2.3) and (2.4)). We have to distinguish three different ranges of $|x|$: long range ($|x| \gg e^{\lambda t}$), medium range ($|x| \sim e^{\lambda t}$), and short range ($|x| \ll e^{\lambda t}$). The proof of (1.4) is stand-alone and presented fully. The proof of (1.5) is nearly identical (and we present it in parallel), but assumes that (1.4) holds a priori.

3.1 Long Range: Kernel Estimates

Lemma 3.1 *For any fixed $c > 0$ and for any x in the range $|x| \geq ce^{\lambda(\alpha+1)t}$, a solution ϕ of (2.3) satisfies*

$$|\phi(x, t)| \leq u(x, t)e^{-\alpha\lambda t}.$$

Later, we will set the value of c based on the parameters of the problem. For now, c only affects the final constant that appears in estimate (3.1). As such, we suppress its notation in the proof of the lemma. We also draw special attention to the fact that, in the long range, we have an explicit exponent for the decay rate of $\lambda\alpha$. This particular value will later facilitate a compatibility condition between the long and medium ranges.

PROOF. We first prove the estimate for ϕ . Set $g(u) = f(u) - \kappa u$. By Duhamel's formula we have

$$\begin{aligned}\phi(x, t) &= e^{\kappa t} \int \partial_{y_i} \rho_\alpha(x - y, t) u_0(y) dy \\ &+ \int_0^t \int e^{\kappa(t-s)} \rho_\alpha(x - y, t - s) \partial_{y_i} g(u(y, s)) dy ds.\end{aligned}\tag{3.2}$$

Note that $\|\partial_x \rho_\alpha(\cdot, t)\|_{L^1}$ is not integrable near $t = 0$, for $\alpha < 1/2$. Hence the second term in the expression for ϕ must be split into two pieces; one in a neighborhood of $s = t$ (where $e^{\kappa(t-s)}$ is essentially constant) and the rest where we may integrate by parts. In total, we obtain the following upper bound on $\phi(x, t)$: $|\phi(x, t)| \leq I_0 + I_1 + I_2$, where

$$\begin{aligned}I_0 &= e^{\kappa t} \int_{\mathbb{R}^d} u_0(x - y) |\partial_y \rho_\alpha(y, t)| dy, \\ I_1 &= \int_0^{t-1} e^{\kappa(t-s)} \int_{\mathbb{R}^d} |f(u(x - y, s)) - \kappa u(x - y, s)| |\partial_y \rho_\alpha(y, t - s)| dy ds, \\ I_2 &= \int_{t-1}^t \int_{\mathbb{R}^d} |f'(u(x - y, s)) \phi(x - y, s) - \kappa \phi(x - y, s)| \rho_\alpha(y, t - s) dy ds.\end{aligned}$$

So we proceed by estimating each of the three integrals. We begin with I_0 and apply the naive estimate for $\partial_x \rho_\alpha$ (2.8) and a concrete decay rate for the initial data u_0 .

$$I_0 \leq e^{\kappa t} P(t) \int_{\mathbb{R}^d} \frac{1}{1 + |x - y|^{d+\alpha+1}} \frac{1}{1 + |y|^{d+\alpha+1}} dy,\tag{3.3}$$

where $P(t)$ is a polynomial in t . The specific form and degree of P are not needed since we will be getting a slight exponential decay in time, and that dominates any residual polynomial growth. To handle the main integral in (3.3), we split into two regions: $\{|y| \leq |x|/2\}$ and $\{|y| > |x|/2\}$. Observe that $|x - y| \geq |x| - |y|$ which, in the first case, means $|x - y| \geq |x|/2$. Thus

$$\int_{|y| \leq |x|/2} \frac{1}{1 + |x - y|^{d+\alpha+1}} \frac{1}{1 + |y|^{d+\alpha+1}} dy \leq \frac{1}{|x|^{d+\alpha+1}} \int_{\mathbb{R}^d} \frac{dy}{1 + |y|^{d+\alpha+1}} \leq \frac{1}{|x|^{d+\alpha+1}}$$

for x in the long range. For the second case,

$$\int_{|y| > |x|/2} \frac{1}{1 + |x - y|^{d+\alpha+1}} \frac{1}{1 + |y|^{d+\alpha+1}} dy \leq \frac{1}{|x|^{d+\alpha+1}} \int_{\mathbb{R}^d} \frac{dy}{1 + |x - y|^{d+\alpha+1}} \leq \frac{1}{|x|^{d+\alpha+1}}.$$

Finally, observe that condition (2.1) and the size of $|x|$ imply that $u(x, t) \approx e^{\kappa t} / |x|^{d+\alpha}$. Since $|x| \geq e^{\lambda(\alpha+1)t}$, we then have

$$I_0 \leq e^{\kappa t} |x|^{-(d+\alpha+1)} \approx u |x|^{-1} \leq u e^{-\lambda(\alpha+1)t},$$

which gives us our desired comparative decay for I_0 .

Continuing, we need to estimate the factors involving f and f' . Since f is smooth enough to have a partial Taylor expansion, we have that $f(u) = f(0) + f'(0)u + O(u^2)$. Therefore $f(u) - \kappa u = O(u^2)$, and this (along with (2.1)) gives one of the upper bounds we will use for I_1 .

$$I_1 \leq \int_0^{t-1} e^{\kappa(t-s)} Q(t, s) \int_{\mathbb{R}^d} \frac{1}{1 + e^{-2\kappa s} |y|^{2(d+\alpha)}} \frac{1}{1 + |x - y|^{d+\alpha+1}} dy ds. \quad (3.4)$$

Here $Q(t, s)$ is algebraic in t and s . As discussed earlier, $Q(t, s)$ will not be integrable up to $s = t$, but it is at most polynomial in t on the interval $0 \leq s \leq t - 1$. As with I_0 , we handle the main integral by splitting into the same two regions: $\{|y| \leq |x|/2\}$ and $\{|y| > |x|/2\}$. In the first case, we get

$$\begin{aligned} & \int_{|y| \leq |x|/2} \frac{1}{1 + e^{-2\kappa s} |y|^{2(d+\alpha)}} \frac{1}{1 + |x - y|^{d+\alpha+1}} dy \\ & \leq \frac{1}{|x|^{d+\alpha+1}} \int_{\mathbb{R}^d} \frac{dy}{1 + e^{-2\kappa s} |y|^{2(d+\alpha)}} \approx \frac{e^{\lambda ds}}{|x|^{d+\alpha+1}}. \end{aligned} \quad (3.5)$$

The last estimate follows from a change of variables. In the second case, we get

$$\begin{aligned} & \int_{|y| > |x|/2} \frac{1}{1 + e^{-2\kappa s} |y|^{2(d+\alpha)}} \frac{1}{1 + |x - y|^{d+\alpha+1}} dy \\ & \leq \frac{e^{2\kappa s}}{|x|^{2(d+\alpha)}} \int_{\mathbb{R}^d} \frac{dy}{1 + |x - y|^{d+\alpha+1}} \approx \frac{e^{2\kappa s}}{|x|^{2(d+\alpha)}}. \end{aligned} \quad (3.6)$$

Plugging the estimates from (3.5) and (3.6) back into (3.4) shows that

$$I_1 \leq \int_0^{t-1} Q(t, s) e^{\kappa(t-s)} \left(\frac{e^{\kappa s \frac{d}{d+\alpha}}}{|x|^{d+\alpha+1}} + \frac{e^{2\kappa s}}{|x|^{2(d+\alpha)}} \right) ds \leq P(t) \left(\frac{e^{\kappa t}}{|x|^{d+\alpha+1}} + \frac{e^{2\kappa t}}{|x|^{2(d+\alpha)}} \right).$$

Remembering that $u \approx e^{\kappa t} |x|^{-(d+\alpha)}$ and $|x| \geq e^{\frac{1+\alpha}{d+\alpha} \kappa t}$, we see that

$$I_1 \leq u P(t) \left(\frac{1}{|x|} + \frac{e^{\kappa t}}{|x|^{d+\alpha}} \right) \leq u P(t) (e^{-\lambda(1+\alpha)t} + e^{-\alpha \kappa t}),$$

establishing the comparative exponential decay rate for I_1 . Observe that our estimates here would not work if we only had $|x| \geq e^{\lambda t}$.

We handle I_2 in an analogous manner. Note that $f'(u) = f'(0) + O(u)$, so that $f'(u)\phi - \kappa\phi = O(u\phi)$. But we also have, from Lemma 2.6,

$$|\phi(x, t)| \leq e^{\kappa t} \rho_\alpha *_{x_i} |\partial_x u_0| \leq C t \frac{e^{\kappa t}}{|x|^{d+\alpha}},$$

and so $|\phi(x, t)| \leq C t u(x, t)$, recalling that Theorem 2.2 implies $u(x, t) \geq C e^{\kappa t} |x|^{-(d+\alpha)}$. This again implies a $O(tu^2)$ upper bound which reduces the estimate on I_2 to

$$I_2 \leq \int_{t-1}^t \bar{Q}(t, s) \int_{\mathbb{R}^d} \frac{1}{1 + e^{-2\kappa s} |y|^{2(d+\alpha)}} \frac{1}{1 + |x - y|^{d+\alpha}} dy ds.$$

Here, $\bar{Q}(t, s)$ is again algebraic in t and s , but can now be made integrable on $[t-1, t]$ due to the correct time-homogeneity of $\rho_\alpha(x, t)$. We estimate the convolution integral exactly as we did for I_1 and find that

$$I_2 \leq \int_{t-1}^t \bar{Q}(t, s) \left(\frac{e^{\kappa s \frac{d}{d+\alpha}}}{|x|^{d+\alpha}} + \frac{e^{2\kappa s}}{|x|^{2(d+\alpha)}} \right) ds \leq P(t) \left(\frac{e^{d\lambda t}}{|x|^{d+\alpha}} + \frac{e^{2\kappa t}}{|x|^{2(d+\alpha)}} \right).$$

Continuing as we did for I_1 , we finally get

$$I_2 \leq uP(t) (e^{d\lambda t - \kappa t} + e^{\kappa t - \kappa(1+\alpha)t}) = uP(t) \left(e^{-\frac{\alpha\kappa t}{d+\alpha}} + e^{-\alpha\kappa t} \right).$$

Finally, we examine the exponents to conclude that, for $|x| \geq e^{\lambda(1+\alpha)t}$, we have $|\phi(x, t)| \leq u(x, t)e^{-\lambda\alpha t}$. Here, again, the method would not have worked if we merely required $|x| \geq e^{\lambda t}$.

Now we turn our attention to ψ . Looking at (2.5), we employ the Duhamel formula to obtain

$$\begin{aligned} \psi(x, t) = & e^{\kappa t} \int \partial_{y_i} \partial_{y_i} \rho_\alpha(x - y, t) u_0(y) dy \\ & + \int_0^t \int e^{\kappa(t-s)} \rho_\alpha(x - y, t - s) \partial_{y_i} g(\phi(y, s)) dy ds \\ & + \int_0^t \int e^{\kappa(t-s)} \rho_\alpha(x - y, t - s) f''(u(y, s)) \phi(y, s)^2 dy ds. \end{aligned} \quad (3.7)$$

The comparative decay estimates follow analogously to the argument for ϕ . The first term has even better decay than I_0 since we placed two derivatives on ρ_α . The second term is again split into two parts \bar{I}_1 and \bar{I}_2 (for the time intervals $[0, t-1]$ and $[t-1, t]$). Since we have already concluded that $|\phi| \leq ue^{-\lambda\alpha t}$ in this range, the analysis for \bar{I}_1 proceeds identically.

Now, $f'(u)\psi - \kappa\psi = O(u\psi)$. The analysis of \bar{I}_2 would also be identical to that of I_2 if we knew that $|\psi| = O(u)$ (up to a time-dependent, not exponentially increasing factor). We prove this by appealing to Lemma 2.6.

$$|\psi(x, t)| \leq e^{\kappa t} \rho_\alpha *_x |\partial_{x_j} \partial_{x_i} u_0| + \int_0^t e^{\kappa(t-s)} \rho_\alpha(\cdot, t-s) *_x |f''(u)\phi\bar{\phi}|(\cdot, s) ds := J_0 + J_1$$

J_0 is bounded by $Cte^{\kappa t}/|x|^{d+\alpha}$ as before. J_1 is estimated by the (now proven) comparative exponential bound for ϕ (and $\bar{\phi}$) in the long range:

$$\begin{aligned} J_1 & \leq Ce^{-2\lambda\alpha t} \int_0^t \int e^{\kappa(t-s)} \rho_\alpha(x - y, t - s) u(y, s)^2 dy ds \\ & \leq Ce^{-2\lambda\alpha t} \int_0^t e^{\kappa(t-s)} \bar{Q}(t, s) \int \frac{1}{1 + e^{-2\kappa s} |y|^{2(d+\alpha)}} \frac{1}{1 + |x - y|^{d+\alpha}} dy ds \\ & \leq P(t) e^{-2\lambda\alpha t} \left(e^{\kappa t} |x|^{-(d+\alpha)} \int_0^t e^{-\alpha\lambda s} ds + e^{\kappa t} |x|^{-2(d+\alpha)} \int_0^t e^{\kappa s} ds \right) \\ & \leq P(t) e^{-2\lambda\alpha t} (u(x, t) + u(x, t)^2). \end{aligned} \quad (3.8)$$

The third inequality was obtained through the same estimates as for I_2 . Notice how J_1 inherits an exponential decay from the more rigorous analysis for ϕ ; using a similar approach for the second term of (3.2) would have merely given us a comparative boundedness for ϕ instead of decay. Hence J_1 is dominated by J_0 . Therefore $\psi = O(tu)$, and we conclude that the second term of (3.7) satisfies a comparative exponential decay rate in the long range.

Lastly, observe that the third term of (3.7) is in fact dominated by J_1 . As seen in (3.8), J_1 also satisfies a comparative exponential decay rate (with exponent $-2\lambda\alpha$). This completes the mirrored argument, demonstrating that $|\psi(x, t)| \leq u(x, t)e^{-\lambda\alpha t}$ as long as $|x| \geq e^{\lambda(1+\alpha)t}$. \bullet

3.2 Short Range: stability of the steady state $u \equiv 1$

The nature of f and condition (2.1) ensures that there exists a fixed $\beta > 0$ such that $\{|\xi| < \frac{3}{2}\beta\} \subseteq \{f'(u) < -\delta_0\}$. That is, we have an exponentially growing ball wherein u is sufficiently close to 1 that $f'(u)$ is negative by at least a fixed amount. Let $\Omega_0(t) = \{|\xi| < \frac{3}{2}\beta\}$. In this region, we use a simple version of the maximum principle. Assume the positive maximum value of ϕ occurs at time \bar{t} at an interior point $\bar{x} \in \Omega_0(\bar{t})$ (negative minimum values are handled similarly). Then

$$\partial_t \phi(\bar{x}, \bar{t}) + \Lambda^\alpha \phi(\bar{x}, \bar{t}) = f'(u(\bar{x}, \bar{t}))\phi(\bar{x}, \bar{t}).$$

However, we have $\Lambda^\alpha \phi(\bar{x}, \bar{t}) \geq 0$. Therefore,

$$\partial_t \phi(\bar{x}, \bar{t}) \leq -\delta_0 \phi(\bar{x}, \bar{t}).$$

We conclude that either ϕ is decaying exponentially (with exponent at least $-\delta_0$) on \mathbb{R}^d , or the size of ϕ on $\Omega_0(t)$ is controlled by the size of ϕ outside $\Omega_0(t)$. Equivalently, v is eventually controlled by a fixed constant times $e^{(\lambda-\delta_0)t}$ whenever $|\xi| < \frac{3}{2}\beta$ so long as v obeys the same bound for (say) $|\xi| > \beta$, which will be established in the next two sections. Certainly, the previous argument shows that this is the case in the long range.

To prove the analogous statement for ψ , we will assume that (1.4) holds with some uniform $\delta > 0$ in all ranges for ϕ (the proof of which will, as stated, be stand-alone). Since u is bounded by 1, we have that

$$\partial_t \psi(x, t) + \Lambda^\alpha \psi(x, t) \leq -\delta_0 \psi(x, t) + Ce^{-2\delta t}$$

for all $x \in \Omega_0(t)$. Assume the positive maximum value of ψ occurs at time \bar{t} at an interior point $\bar{x} \in \Omega_0(\bar{t})$. Then

$$\partial_t \psi(\bar{x}, \bar{t}) \leq -\delta_0 \psi(\bar{x}, \bar{t}) + Ce^{-2\delta \bar{t}}$$

again establishing that *either* ψ is decaying exponentially on \mathbb{R}^d , *or* the size of ψ on $\Omega_0(t)$ is controlled by the size of ψ outside $\Omega_0(t)$. As before, negative minima are treated similarly.

3.3 Medium range: selective comparison principle

For the intermediate range, we appeal to Theorem (2.4). Let $\Omega_1(t) = \{\beta \leq |\xi| \leq ce^{\lambda\alpha t}\}$, for $c > 0$ to be determined later. We need to find a w such that $w(\xi, 0) \geq |v(\xi, 0)|$, and $w > |v|$ outside of $\Omega_1(t)$ for all $t > 0$, and such that

$$\partial_t w - \lambda \xi \cdot \nabla w + e^{-\lambda\alpha t} \Lambda^\alpha w - \lambda w - f'(u)w \geq 0 \quad (3.9)$$

pointwise on $\Omega_1(t)$. It is important that w be defined on all of \mathbb{R}^d because Λ^α is nonlocal; while it can be defined on compact sets, it becomes a different operator than the one in (2.4).

We must therefore provide a candidate supersolution w . For $\xi \in \Omega_1(t)$ and $0 < \nu - d - \alpha \leq 1$, define the family of functions \tilde{w}_ν by

$$\tilde{w}_\nu(\xi, t) = \frac{\beta^\nu w_0}{|\xi|^\nu} + C_\nu e^{-\lambda\alpha t} \left(\frac{\beta^{\nu-d-\alpha}}{|\xi|^\nu} - \frac{1}{|\xi|^{d+\alpha}} \right), \quad (3.10)$$

where $C_\nu, w_0 > 0$ will be determined later. Observe that each such \tilde{w}_ν is radial. To simplify notation, let $y = |\xi|$. For fixed ν and t , \tilde{w}_ν is the solution of the following ODE in y (on $\Omega_1(t)$):

$$y \frac{d}{dy} \tilde{w}_\nu(y, t) + \nu \tilde{w}_\nu(y, t) = -\bar{C}_\nu e^{-\lambda\alpha t} y^{-(d+\alpha)}. \quad (3.11)$$

Note that $C_\nu = \bar{C}_\nu / (\nu - d + \alpha)$. \bar{C}_ν will be determined below. Technically, \bar{C}_ν will also depend on w_0 , but the dependence will be linear. More importantly, this dependence is not circular: w_0 is the initial condition at $y = \beta$, and this may be fixed before solving the ODE. w_0 itself will depend on the size of $v(\xi, 0)$.

Our supersolution must be positive, yet \tilde{w}_ν is eventually negative for sufficiently large y . We now choose c such that $\tilde{w}_\nu > 0$ on $\Omega_1(t)$; since $\nu \leq d + \alpha + 1$, such a choice is always possible. Outside of $\Omega_1(t)$, we extend \tilde{w}_ν to be smooth, bounded, positive, radial, decreasing in y , and $O(y^{-(d+\alpha)})$. This can obviously be done such that the L^1 , C^1 , and C^2 norms of \tilde{w} are uniformly bounded for all $t > 0$. They do, however, depend on ν and w_0 .

We now define our candidate supersolution as $w(\xi, t) = e^{\eta t} \tilde{w}_\nu(|\xi|, t)$ with η and ν fixed; we will determine their values shortly. Observe that $\xi \cdot \nabla w = y e^{\eta t} \partial_y \tilde{w}$. To prove (3.9), it therefore suffices to show that

$$-e^{\eta t} \partial_t \tilde{w}_\nu - \eta w + \lambda y \partial_y w(y, t) + (\lambda + f'(u(\xi, t))) w(y, t) \leq -e^{-\lambda\alpha t} |\Lambda^\alpha w(y, t)|$$

on $\Omega_1(t)$. Examining (3.10), we see that $\partial_t \tilde{w}_\nu > 0$ on our domain, so we get a stronger inequality if we ignore the first term above; proving the stronger inequality will be sufficient. Since $f'(u) \leq \kappa = \lambda(d + \alpha)$ and $w > 0$, we can eliminate the ξ dependence entirely and satisfy the above inequality provided that

$$\lambda y \partial_y w(y, t) + \lambda(d + \alpha + 1 - \frac{\eta}{\lambda})w(y, t) \leq -e^{-\lambda \alpha t} |\Lambda^\alpha w(y, t)|.$$

Treating the diffusion as a perturbation term, we need to bound the size of $|\Lambda^\alpha w(y, t)|$. Our choice of extension for \tilde{w}_ν onto \mathbb{R}^d implies (see [3], and estimate (8) of [6]):

$$|\Lambda^\alpha w(y, t)| = e^{\eta t} |\Lambda^\alpha \tilde{w}_\nu(y, t)| \leq e^{\eta t} \frac{\bar{C}_\nu}{y^{d+\alpha}}$$

on $\Omega_1(t)$. Our assumptions on the uniform-in-time bounds of the norms of \tilde{w}_ν ensure that the constant \bar{C}_ν can be made time-independent (recall that ν is fixed). It is also linear in w_0 , as mentioned before. Thus, proving (3.9) reduces to showing that

$$y \partial_y w(y, t) + (d + \alpha + 1 - \frac{\eta}{\lambda})w(y, t) = -e^{\eta t - \lambda \alpha t} \frac{\bar{C}_\nu}{y^{d+\alpha}},$$

with $\bar{C}_\nu > 0$ now determined. But this is precisely (3.11), multiplied by $e^{\eta t}$ with $\nu = d + \alpha + 1 - \frac{\eta}{\lambda}$. Hence, (3.9) holds on $\Omega_1(t)$. Clearly, we need $0 \leq \eta < \lambda$. We then choose w_0 sufficiently large that w (that is, the function extended to all of \mathbb{R}^d) dominates $|v|$ at $t = 0$.

In order to use Theorem (2.4), we need to show that $w - |v| > 0$ outside of $\Omega_1(t)$. It is here that we needed our free parameters ν and η ; naively, we would like to take $\eta = 0$ and $\nu = d + \alpha + 1$, as this would give the best bounds in the medium range. However, the upper bounds for ϕ established in the previous two sections are too weak to ensure that $w - |v| > 0$ outside of $\Omega_1(t)$ for all t with that naive choice. In the long range, we only have $|v(\xi, t)| < e^{\lambda(1-\alpha)t} |\xi|^{-(d+\alpha)}$. In the short range, we have that $|v(\xi, t)| < e^{(\lambda-\delta_0)t}$ provided the same can be said of v globally.

The added growth rate (in the form of $e^{\eta t}$) allows us to ensure that $w > |v|$ in the long range. Since both w and the upper bound for v are radially decreasing and $O(|\xi|^{-(d+\alpha)})$, it suffices to check the inequality for $|\xi| = ce^{\lambda \alpha t}$. At the boundary between medium range and long range, $w \approx e^{-\nu \lambda \alpha t + \eta t}$ and $|v| < e^{-(d+\alpha)\lambda \alpha t + \lambda t - \lambda \alpha t}$. So it suffices to have $\eta - \lambda \alpha(\nu - d - \alpha) = \eta - \lambda \alpha(1 - \frac{\eta}{\lambda}) \geq \lambda(1 - \alpha)$, or $\eta \geq \frac{\lambda}{1+\alpha}$. This leaves enough room for η to be smaller than λ . Fixing constants (and looking at the equation after a transient period to allow the long range decay estimates to come into effect) we have that $w > |v|$ for $|\xi| > ce^{\lambda \alpha t}$.

For the second derivatives, we will similarly employ a supersolution of the form $W(\xi, t) = e^{\zeta t} \tilde{w}_\nu(|\xi|, t)$ with nearly the same \tilde{w}_ν as in (3.10); specifically, we require a larger constant \bar{C}_ν in equation (3.11). Essentially, we need W to satisfy

$$-\partial_t W + \lambda \xi \cdot \nabla W + 2\lambda W + f'(u)W \leq -e^{-\lambda \alpha t} |\Lambda^\alpha W| - Ce^{2\lambda t - 2\delta t} u^2, \quad (3.12)$$

once again appealing to the full estimate for ϕ . As before, $|\Lambda^\alpha W(\xi, t)| \leq \bar{C}_\nu e^{\zeta t} |\xi|^{-(d+\alpha)}$. But the last term is dominated by $Ce^{2\lambda - 2\delta t} |\xi|^{-2(d+\alpha)}$. If we assume that $\zeta > 2\lambda - 2\delta$, the diffusive term is then a stronger perturbation than the

inhomogeneity, and so the latter can be absorbed into the bounds for the former. Our candidate supersolution W will solve (3.12) provided that

$$y\partial_y\tilde{w}_\nu(y, t) + (d + \alpha + 2 - \frac{\zeta}{\lambda})\tilde{w}_\nu(y, t) \leq \frac{\bar{C}_\nu}{y^{d+\alpha}}.$$

This is just (3.11) with $\nu = d + \alpha + 2 + \zeta/\lambda$, so the equation is satisfied. We want to take ζ as small as possible while still observing the requirement that $0 < \nu - d - \alpha \leq 1$ (this is necessary to ensure W remains positive on $\Omega_1(t)$ with uniform bounds). Thus our arguments impose the constraint that $2\lambda - 2\delta < \zeta < 2\lambda$.

For the final constraint, W must dominate V (that is, $e^{2\lambda t}\psi$ in exponential coordinates) at the interface between medium and long ranges. As before, we have that $W \approx e^{-\nu\lambda\alpha t + \zeta t}$ and $|V| \leq e^{-(d+\alpha)\lambda\alpha t + 2\lambda t - \lambda\alpha t}$ when $|\xi| = ce^{\lambda\alpha t}$. We therefore require that $\zeta - \lambda\alpha(\nu - d - \alpha) = \zeta - \lambda\alpha(2 - \frac{\zeta}{\lambda}) \geq \lambda(2 - \alpha)$, or $\zeta \geq \lambda\frac{2+\alpha}{1+\alpha}$. For $\alpha > 0$, this will always overlap with the admissible range of ζ determined above; and, as expected, the minimum value for ζ decreases as we get better diffusion.

This establishes, as before, that $W > |V|$ in the long range and that W satisfies the comparison inequality on $\Omega_1(t)$. To satisfy all the conditions for the selective comparison principle, we must show that the supersolutions for v and V also dominate in the short range, $\Omega_0(t)$.

3.4 The estimate for ϕ and ψ

PROOF OF THEOREM 1.2. Recall that $\Omega_0(t) = \{|\xi| < \frac{3}{2}\beta\}$ and $\Omega_1(t) = \{\beta < |\xi| < ce^{\lambda\alpha t}\}$. Section 3.2 established that, in $\Omega_0(t)$, $v < C(v_0)e^{(\lambda-\delta_0)t}$ or else v is larger outside of this regime; which, by Lemma 3.1, in fact means v is larger in $\Omega_1(t)$; outside of $\Omega_0(t) \cup \Omega_1(t)$ (a compact set), the comparative exponential bound holds unconditionally.

Fix $\eta = \max(\frac{\lambda}{1+\alpha}, \lambda - \delta_0/2)$ and w_0 so large that the conditional bound in the short range is stronger than the requirement that $w > v$ in the same regime; that is, $C(v_0)e^{(\lambda-\delta_0)t} < \inf_{\xi \in \Omega_0(t)} w(\xi, t)$ for all $t > 0$ ((3.10) shows that such a choice is always possible). We can also ensure that $w > v$ at $t = 0$.

We then argue by contradiction. Let $\bar{t} > 0$ be the critical time such that $v(\xi, t) < w(\xi, t)$ for all $t < \bar{t}$ and $\xi \in \Omega_0(t) \setminus \Omega_1(t)$, and such that the same upper bound fails for a sequence $t_n \rightarrow \bar{t}^+$. This therefore implies a sequence $\{\xi_n\} \in \Omega_0(t_n) \setminus \Omega_1(t_n)$ such that $v(\xi_n, t_n) \geq w(\xi_n, t_n) \geq C(v_0)e^{(\lambda-\delta_0)t}$. By the conditional bound of Section 3.2, there is another sequence $\{\xi'_n\} \in \Omega_1(t_n) \setminus \Omega_0(t_n)$ such that $v(\xi'_n, t_n) \geq \inf_{\xi \in \Omega_0(t_n) \setminus \Omega_1(t_n)} w(\xi, t_n)$. By compactness and the continuity of v , we extract a limit point ξ with $|\xi| \geq \frac{3}{2}\beta$ and $v(\xi, \bar{t}) \geq \inf_{|\xi| < \beta} w(\xi, \bar{t})$.

However, for $t < \bar{t}$, the hypothesis of Theorem (2.4) was by assumption valid, and therefore $v < w$ up to \bar{t} . Moreover, (3.10) immediately shows that there is a

continuous, *always positive* $\varepsilon(t)$ such that $\sup_{|\xi| \geq \frac{3}{2}\beta} w(\xi, t) < \inf_{|\xi| < \beta} w(\xi, t) - \varepsilon(t)$. The continuity of v allows us to extend the inequality to \bar{t} . Thus

$$v(\bar{\xi}, \bar{t}) \geq \inf_{|\xi| < \beta} w(\xi, \bar{t}) \geq \sup_{|\xi| \geq \frac{3}{2}\beta} w(\xi, \bar{t}) + \varepsilon(\bar{t}) \geq w(\bar{\xi}, \bar{t}) + \varepsilon(\bar{t}) \geq v(\bar{\xi}, \bar{t}) + \varepsilon(\bar{t}),$$

which is evidently a contradiction. Therefore, $w > v$ (and analogously $w > -v$) in $\Omega_0(t) \setminus \Omega_1(t)$. The hypothesis of Theorem (2.4) is always satisfied, and so $w(\xi, t) > |v(\xi, t)|$ for all ξ and t . Precisely the same argument also applies to V with $\zeta = \max(\lambda \frac{2+\alpha}{1+\alpha}, 2\lambda - 2\delta, 2\lambda - \delta_0)$.

Putting everything together, we have the following three upper bounds (written in terms of ϕ). Recall that $u \approx |\xi|^{-(d+\alpha)}$ in the medium range.

$$\begin{aligned} |\phi(x, t)| &\leq C_2 u(x, t) e^{-\lambda \alpha t} \text{ for } |x| > c_1 e^{\lambda(1+\alpha)t} \\ |\phi(x, t)| &\leq C_1 u(x, t) e^{\eta t - \lambda t} |\xi|^{-1+\eta/\lambda} \leq C_1 u(x, t) e^{-\delta_1 t} \text{ for } c_0 e^{\lambda t} \leq |x| \leq c_1 e^{\lambda(1+\alpha)t} \\ |\phi(x, t)| &\leq C_0 e^{-\delta_0 t} \leq C_0 u(x, t) e^{-\delta_0 t} \text{ for } |x| < c_0 e^{\lambda t} \end{aligned}$$

We also get the corresponding three bounds for ψ (recall that $\psi = e^{-2\lambda t} V$).

$$\begin{aligned} |\psi(x, t)| &\leq C'_2 u(x, t) e^{-\lambda \alpha t} \text{ for } |x| > c_1 e^{\lambda(1+\alpha)t} \\ |\psi(x, t)| &\leq C'_1 u(x, t) e^{\zeta t - 2\lambda t} |\xi|^{-2+\zeta/\lambda} \leq C'_1 u(x, t) e^{-\delta'_1 t} \text{ for } c_0 e^{\lambda t} \leq |x| \leq c_1 e^{\lambda(1+\alpha)t} \\ |\psi(x, t)| &\leq C_0 e^{-\delta_0 t} \leq C_0 u(x, t) e^{-\delta_0 t} \text{ for } |x| < c_0 e^{\lambda t} \end{aligned}$$

Thus proving (3.1) and Theorem (1.2). •

Remark: Theorem (1.2) can be extended to a comparative exponential decay for *all* derivatives of u . The dual proof presented above is inductive, essentially repeating the same arguments for ψ while assuming the full result for ϕ . This can be reworked into a full induction showing the much stronger result that

$$|\nabla^k u(x, t)| \leq C u(x, t) e^{-\delta_k t}$$

for all k . The next section will, however, only require (3.1).

4 Bound on the fractional Laplacian

We finally come to the estimate telling us that, at large times, the fractional Laplacian of the solution u is small compared to the size of u .

Lemma 4.1 *There exist $C > 0$ and $\delta' > 0$ such that*

$$|\Lambda^\alpha u(x, t)| \leq C u(x, t) e^{-\delta' t}. \tag{4.1}$$

PROOF. We will examine the explicit integral representation for $\Lambda^\alpha u(x, t)$ and initially with $|x| \geq ce^{\lambda t}$. Up to a constant, we have

$$\begin{aligned}\Lambda^\alpha u(x, t) &= \left(\int_{|x-y| < \varepsilon e^{\gamma \lambda t}} dy + \int_{|x-y| \geq \varepsilon e^{\gamma \lambda t}} dy \right) \frac{u(x, t) - u(y, t)}{|x-y|^{d+\alpha}}, \\ &= I_1 + I_2\end{aligned}\quad (4.2)$$

where $\gamma \in (0, 1]$. For the inner piece I_1 , we make use of (3.1) and a Taylor expansion: $u(x, t) - u(y, t) = -\nabla u(x, t) \cdot (y - x) + O(|\nabla^2 u(cy + (1-c)x, t)||x-y|^2)$ for some $c \in (0, 1)$. Keeping in mind that

$$\int_A \frac{x-y}{|x-y|^{d+\alpha}} dy = 0$$

on any annulus A centered at x , we then have

$$|I_1| \leq \int_{|x-y| < \varepsilon e^{\gamma \lambda t}} \frac{|\nabla^2 u(cx + (1-c)y, t)|}{|x-y|^{d+\alpha-2}} dy \leq e^{-\delta t} \int_{|x-y| < \varepsilon e^{\gamma \lambda t}} \frac{u(cx + (1-c)y, t)}{|x-y|^{d+\alpha-2}} dy.$$

Now, in this regime, $u(x, t) \approx e^{\kappa t} |x|^{-(d+\alpha)}$, and $|cx + (1-c)y| \geq (1-\varepsilon)|x|$ because we assumed $|x-y| < \varepsilon e^{\gamma \lambda t} \leq \varepsilon |x|^\gamma \leq \varepsilon |x|$. Therefore,

$$|I_1| \leq (1-\varepsilon)^{-(d+\alpha)} u(x, t) e^{-\delta t} \int_0^{\varepsilon e^{\gamma \lambda t}} \frac{r^{d-1} dr}{r^{d-\alpha-2}} dr.$$

We then have (up to a constant depending on ε)

$$|I_1| \leq u(x, t) e^{-\delta t} e^{\gamma \lambda (2-\alpha)t} \leq u(x, t) e^{-\delta t/2}. \quad (4.3)$$

To obtain the last inequality we must choose γ appropriately. If $(2-\alpha)\lambda \leq \delta/2$, we may take $\gamma = 1$. Otherwise, we take $\gamma = \delta/(2\lambda(2-\alpha)) < 1$. Either way, we obtain our final estimate for I_1 .

For the outer piece I_2 , we merely use the positivity of u : $(u(x, t) - u(y, t)) < u(x, t)$.

$$|I_2| \leq u(x, t) \int_{|x-y| \geq \varepsilon e^{\gamma \lambda t}} \frac{dy}{|x-y|^{d+\alpha}} = u(x, t) \int_{\varepsilon e^{\gamma \lambda t}}^\infty \frac{r^{d-1} dr}{r^{d+\alpha}} = u(x, t) \varepsilon^{-\alpha} e^{-\alpha \gamma \lambda t}.$$

The above inequality and (4.3) show that $|\Lambda^\alpha u(x, t)| \leq u(x, t) e^{-\delta' t}$ for some fixed positive δ' and $|x| > ce^{\lambda t}$. The exact value (in terms of δ , α , and λ) requires an optimization argument for γ .

The case where $|x| < ce^{\lambda t}$, i.e. the short range, is much simpler. In this range, we have that $0 < \bar{c} < u(x, t) \leq 1$. And, moreover, we know from Theorem 1.2 that $\|\nabla^2 u\|_{L^\infty} \leq Ce^{-\delta t}$. Using our Taylor expansion again, we can interpolate between these two to obtain:

$$\left| \int_{\mathbb{R}^d} \frac{u(x, t) - u(y, t)}{|x-y|^{d+\alpha}} dy \right| \leq \int_{|x-y| < R} \frac{\|\nabla^2 u(\cdot, t)\|_{L^\infty} dy}{|x-y|^{d+\alpha-2}} + \int_{|x-y| \geq R} \frac{2dy}{|x-y|^{d+\alpha}} \leq Ce^{-\alpha \delta t/2}.$$

The last inequality required us to optimize R to $\|\nabla^2 u(\cdot, t)\|_{L^\infty}^{-1/2}$ and use our uniform estimate. Hence $|\Lambda^\alpha u(x, t)| \leq Ce^{-\delta' t} \leq C\bar{c}^{-1} u(x, t) e^{-\delta' t}$ for all x . \bullet

5 Asymptotic symmetrization

We have used (3.1) to establish a similar comparative decay on the asymptotic size of $\Lambda^\alpha u$. We will now look at the explicit (integral) representation for the solution $u(x, t)$ and obtain a better approximate asymptotic behavior than what may be inferred from (2.1). We will then use this to prove Theorem 1.3.

5.1 Behavior of u at infinity for $t = 1$

The second ingredient in the symmetrization proof is the fact that $u(x, 1)$ decays exactly as $|x|^{-(d+\alpha)}$ for large x . This is detailed in the following lemma.

Lemma 5.1 *For all $t > 0$, there is $k_t > 0$ and $\delta > 0$ such that*

$$u(x, t) = \frac{k_t}{|x|^{(d+\alpha)}} + O\left(\frac{1}{|x|^{d+\alpha+\delta}}\right). \quad (5.1)$$

PROOF. Once again it is a matter of applying Duhamel's formula to u :

$$\begin{aligned} u(x, t) &= \frac{e^{\kappa t}}{t^{d/\alpha}} \int_{\mathbb{R}^d} p_\alpha\left(\frac{x-y}{t^{1/\alpha}}\right) u_0(y) dy + \int_0^t \int_{\mathbb{R}^d} \frac{e^{\kappa(t-s)}}{(t-s)^{d/\alpha}} p_\alpha\left(\frac{x-y}{(t-s)^{1/\alpha}}\right) u^2(y, s) dy \\ &= \frac{e^{\kappa t}}{t^{d/\alpha}} \int_{\mathbb{R}^d} p_\alpha\left(\frac{x-y}{t^{1/\alpha}}\right) u_0(y) dy + \int_0^t e^{\kappa(t-s)} D(s, t, x, y) ds. \end{aligned}$$

We have, by Proposition 2.3:

$$\begin{aligned} \int_{\mathbb{R}^d} p_\alpha\left(\frac{x-y}{t^{1/\alpha}}\right) u_0(y) dy &= c_\alpha t \int_{\mathbb{R}^d} \frac{u_0(y)}{|x-y|^{d+\alpha}} dy + O\left(\frac{1}{|x|^{d+\alpha+\delta}}\right) \\ &= \frac{c_\alpha t}{|x|^{d+\alpha}} \int_{\mathbb{R}^d} u_0 dy + O\left(\frac{1}{|x|^{d+\alpha+\delta}}\right) \end{aligned}$$

because u_0 is exponentially decreasing. As for the second term, it is sufficient to estimate $D(s, t, x, y)$, the integration in time being only between 0 and t . Let us decompose the integral

$$D(s, t, x, y) = \int_{|x-y| \leq |x|/2} + \int_{|x-y| \geq |x|/2} := I_1 + I_2.$$

We omit, in order to keep the notations light, the dependence in s, t, x, y . As for I_1 , let us notice the bound $u(x, t) \leq \frac{C e^{\kappa t}}{|x|^{d+\alpha}}$ simply because we have $\partial_t u + \Lambda^\alpha u \leq \kappa u$. So, we have

$$\begin{aligned} I_1 &\leq \frac{C_t}{|x|^{2(d+\alpha)} (t-s)^{d/\alpha}} \int_{|x-y| \leq |x|/2} p_\alpha\left(\frac{x-y}{(t-s)^{1/\alpha}}\right) dy \\ &\leq \frac{C_t}{|x|^{2(d+\alpha)}} \|p_\alpha\|_{L^1}. \end{aligned}$$

As for I_2 , let us use estimate (2.9) in Proposition 2.3 and write $I_2 \leq I_{21} + CI_{22}$, where

$$I_{22} = (t-s)^{1+\delta/\alpha} \int_{|x-y| \geq |x|/2} \frac{u^2(y, s)}{|x-y|^{d+\alpha+\delta}} dy \leq \frac{C_t}{|x|^{d+\alpha+\delta}},$$

and

$$I_{21} = (t-s) \int_{|x-y| \geq |x|/2} \frac{u^2(y, s)}{|x-y|^{d+\alpha}} dy.$$

Let us split I_{21} as

$$I_{21} = \int_{|x-y| \geq |x|/2, |y| \leq \varepsilon|x|} + \int_{|x-y| \geq |x|/2, |y| \geq \varepsilon|x|} = I_{211} + I_{212},$$

where $\varepsilon > 0$ is small. We have $I_{212} \leq \frac{C_{t,\varepsilon}}{|x|^{2(d+\alpha)}}$, so it remains to study the last term.

We use the fact that, for $|y| \leq \varepsilon|x|$ we have

$$|x-y| = |x|(1 + O\left(\frac{|y|}{|x|}\right));$$

in other words we have, setting

$$C_x := \{|x-y| \geq |x|/2, |y| \leq \varepsilon|x|\}$$

and using once again estimate (2.9):

$$\begin{aligned} I_{211} &= \frac{c_\alpha(t-s)}{|x|^{d+\alpha}} \int_{C_x} u^2(y, s) dy + O\left(\frac{t-s}{|x|^{d+\alpha+1}} \int_{C_x} |y| u^2(y, s) dy\right) \\ &\quad + O\left(\frac{(t-s)^{1+\delta/\alpha}}{|x|^{d+\alpha+\delta}} \int_{C_x} u^2(y, s) dy\right). \end{aligned}$$

Set $l(t) = \int_{\mathbb{R}^d} u^2(y, t)$ and note that $\int_{\mathbb{R}^d} |y| u^2(y, t) dy < +\infty$. Hence there is $l_s > 0$ such that

$$I_{211} = \frac{c_\alpha(t-s)l_s}{|x|^{d+\alpha}} + O\left(\frac{1}{|x|^{d+\alpha+\delta}}\right).$$

Gathering everything yields (5.1). •

5.2 Analysis of long-time level sets

The proof of Theorem 1.3 now reduces to the explicit resolution of a simple ODE.

PROOF OF THEOREM 1.3. For all $\lambda \in [0, 1)$, there exists $m_\lambda > 0$ such that

$$\forall u \in [0, \lambda], \quad m_\lambda u \leq f(u)/\kappa \leq u.$$

The combination of this remark and Lemma 4.1 reduces the full problem (1.1) to

$$\partial_t u = (1 + O(e^{-\delta t}))f(u), \quad u(x, 1) = \frac{k_1}{|x|^{d+\alpha}} + O\left(\frac{1}{|x|^{d+\alpha+\delta}}\right) \text{ as } |x| \rightarrow +\infty, \quad (5.2)$$

keeping in mind that $O(e^{-\delta t})$ might depend on λ . Set

$$G(u) = \int \frac{du}{f(u)},$$

as $u \rightarrow 0$ we have

$$G(u) = \frac{\text{Log} u}{\kappa} + g_0 + O(u),$$

where g_0 is a positive constant. Finding the λ -level set of u for large t amounts to integrating (5.2) with x large, which yields

$$\int_1^t \frac{\partial_t u}{f(u)} = 1 + O(e^{-\delta t}),$$

in other words

$$\text{Log} u + O(u) - \text{Log} \left(\frac{k_1}{|x|^{d+\alpha}} + O\left(\frac{1}{|x|^{d+\alpha+\delta}}\right) \right) = \kappa t + q_\infty(u_0) + O(e^{-\delta t}).$$

Specializing $u(x, t) = \lambda$ yields

$$\exp \left(O\left(\frac{1}{|x|^{d+\alpha}}\right) \right) \left(\frac{k_1}{|x|^{d+\alpha}} + O\left(\frac{1}{|x|^{d+\alpha+\delta}}\right) \right) = e^{-\kappa t + q_{\lambda, \infty}(u_0) + O(e^{-\delta t})},$$

for some possibly different constant $q_{\lambda, \infty}(u_0)$. This, after elementary computations, yields (1.6). •

Let us point out again that we have proved, in quite a strong sense, that the large time dynamics of the KPP problem (1.1) is the same as that of the ODE $\dot{u} = f(u)$, the fractional diffusion being there only, but this is important to set the initial datum right. This fact was already noticed in [6] and, in an even more evident fashion, in [12].

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